

The complex Monge-Ampère equation on some compact Hermitian manifolds

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Abstract

We consider the complex Monge-Ampère equation on compact manifolds when the background metric is a Hermitian metric (in complex dimension two) or a kind of Hermitian metric (in higher dimensions). We prove that the Laplacian estimate holds when F is in W^{1,q_0} for any $q_0 > 2n$. As an application, we show that, up to scaling, there exists a unique classical solution in W^{3,q_0} for the complex Monge-Ampère equation when F is in W^{1,q_0} .

1 Introduction

We consider the regularity problem of the complex Monge-Ampère equation on some compact Hermitian manifolds. Let (M, g) be a compact Hermitian manifold of complex dimension $n \geq 2$. For a real-valued function F on M , we consider the Monge-Ampère equation

$$\det(g_{i\bar{j}} + \phi_{i\bar{j}}) = e^F \det(g_{i\bar{j}}),$$

with $(g_{i\bar{j}} + \phi_{i\bar{j}}) > 0$, for a real-valued function ϕ such that $\sup_M \phi = -1$. We write

$$\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

and

$$\tilde{\omega} = \sqrt{-1} \tilde{g}_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

where $\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \phi_{i\bar{j}}$. Thus, the Monge-Ampère equation can be written as

$$\begin{cases} \tilde{\omega}^n = e^F \omega^n \\ \tilde{\omega} = \omega + \sqrt{-1} \partial \bar{\partial} \phi > 0 \\ \sup_M \phi = -1 \end{cases} \quad (1.1)$$

We shall use the following notations, for a function f and a holomorphic coordinate $z = (z^1, \dots, z^n)$,

$$f_{i\bar{j}} = \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j}, \quad \Delta f = g^{i\bar{j}} f_{i\bar{j}}, \quad \tilde{\Delta} f = \tilde{g}^{i\bar{j}} f_{i\bar{j}},$$

$$|\nabla f|^2 = g^{i\bar{j}} f_i f_{\bar{j}}, \quad |\tilde{\nabla} f|^2 = \tilde{g}^{i\bar{j}} f_i f_{\bar{j}}.$$

What is more, we use $\|f\|_{L^p(M, \omega)}$ and $\|\nabla^m f\|_{L^p(M, \omega)}$ to denote the corresponding norms with respect to (M, ω) .

When ω is Kähler, the complex Monge-Ampère equation is very important. In the 1950s, Calabi [8] presented his famous conjecture and transformed that problem into (1.1). In [31], Yau proved

the existence of the classical solution of (1.1) by using the continuity method and solved the Calabi's conjecture.

The Dirichlet problem for the complex Monge-Ampère equation is also very important. On one hand, Bedford-Taylor [6, 7] studied the weak solution. After their work, weak solution of the complex Monge-Ampère equation has been studied extensively. There are many existence, uniqueness and regularity results of the complex Monge-Ampère equation under different conditions and we refer the reader to [4, 12, 13, 14, 18, 20, 21, 32].

On the other hand, the classical solvability of the Dirichlet problem was established by Caffarelli-Kohn-Nirenberg-Spruck [11] for strongly pseudoconvex domains in \mathbb{C}^n . The reader can also see the work of Krylov [22, 23]. For further information, we refer the reader to [25] which is a survey of some recent developments in the theory of complex Monge-Ampère equation.

When ω is not Kähler, the existence of the solution of the complex Monge-Ampère equation has been studied under some assumptions on ω (see [9, 16, 19, 27]). For a general ω , Tosatti-Weinkove [28] has gotten the key C^0 -estimate. As an application, they have showed that, up to scaling, the complex Monge-Ampère equation on a compact Hermitian manifold admits a smooth solution when the right hand side F is smooth.

In [10], Chen-He have proved that, on a compact Kähler manifold of complex dimension n , the Laplacian estimate and the gradient estimate hold and there exists a classical solution in W^{3,q_0} for the complex Monge-Ampère equation when the right hand side F is in W^{1,q_0} for any $q_0 > 2n$.

In this paper, we generalize the work of Chen-He [10]. We use a different method (we don't need the gradient estimate to get the Laplacian estimate) to consider the regularity problem of (1.1) on some compact Hermitian manifolds (including compact Kähler manifolds).

We introduce a definition first.

Definition 1.1. Let (M, ω) be a compact Hermitian manifold of complex dimension n , if for any

$$\phi \in \{\varphi \in C^2(M) \mid \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0, \|\varphi\|_{L^\infty(M, \omega)} \leq \Lambda_1 \text{ and } -\Lambda_2\omega^n \leq \tilde{\omega}^n \leq \Lambda_2\omega^n\},$$

there exists a constant $C = C(\Lambda_1, \Lambda_2, M, \omega)$, such that

$$-C\omega^n \leq \sqrt{-1}\partial\bar{\partial}\tilde{\omega}^{n-1} \leq C\omega^n.$$

Then, we say (M, ω) satisfies condition (*).

Remark 1.2. When $n = 2$, condition (*) is trivial. Since

$$\partial\bar{\partial}\tilde{\omega} = \partial\bar{\partial}\omega,$$

all compact Hermitian manifolds of complex dimension 2 satisfy condition (*).

Remark 1.3. When $n = 3$, if (M, ω) is a compact Hermitian manifold satisfies

$$\partial\bar{\partial}\omega = 0,$$

then we have

$$\partial\bar{\partial}\tilde{\omega}^2 = 2\partial\omega \wedge \bar{\partial}\omega,$$

which implies this Hermitian manifold (M, ω) satisfies condition (*).

Remark 1.4. When $n \geq 4$, Condition $(*)$ is not a very strong restricted condition. For example, if (M, ω) is a compact Hermitian manifold satisfies

$$\partial\bar{\partial}\omega = 0 \quad \text{and} \quad \partial\bar{\partial}\omega^2 = 0. \quad (1.2)$$

Then we can conclude that $\partial\bar{\partial}\omega^k = 0$ for all $1 \leq k \leq n-1$ (see, for example, [15]), which implies $\partial\bar{\partial}\tilde{\omega}^k = 0$ for all $1 \leq k \leq n-1$. Thus, such Hermitian manifold (satisfying (1.2)) satisfies condition $(*)$. For example, the product of a complex curve with a Kähler metric and a complex surface with a non-Kähler Gauduchon metric satisfies (1.2). More examples are constructed in [15].

Remark 1.5. All compact Kähler manifolds satisfy condition $(*)$.

Now, we state our Laplacian estimate as follows.

Theorem 1.6. *Let (M, ω) be a compact Hermitian manifold of complex dimension n . Assume that either*

(1) $n = 2$; or

(2) $n \geq 3$ and (M, ω) satisfies condition $(*)$.

If ϕ is a smooth solution of (1.1), then

$$\|n + \Delta\phi\|_{L^\infty(M, \omega)} \leq C(\|F\|_{W^{1, q_0}(M, \omega)}, q_0, M, \omega).$$

Usually, we need the gradient estimate to derive the Laplacian estimate. However, the computation on Hermitian manifolds is more complicated due to the existence of torsion terms. As a result, the gradient estimate is very difficult to get. In order to solve this problem, we introduce a new method to get the Laplacian estimate directly. By using Moser's iteration (see [24]), L^p estimates (for example, see [17]) and some interpolation inequalities, we can obtain the Laplacian estimate without doing any calculation about the gradient, which makes the argument more simple and clear. Therefore, we believe that our ideas can be applied to other nonlinear equations on compact manifolds.

As an application of Theorem 1.6, we have the following theorem.

Theorem 1.7. *Assume that (M, ω) satisfies (1) or (2) in Theorem 1.6. Let F be a function in W^{1, q_0} for any $q_0 > 2n$. Then, there exist a function $\phi \in W^{3, q_0}$ and a constant b , such that*

$$\begin{cases} \tilde{\omega}^n = e^{F+b\omega} \omega^n \\ \tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0 \\ \sup_M \phi = -1 \end{cases}.$$

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2 Some preliminary computations

We need the following C^0 -estimate from [28].

Theorem 2.1. *For any compact Hermitian manifold (M, ω) , if ϕ is a smooth solution of (1.1), then we have*

$$\|\phi\|_{L^\infty(M, \omega)} \leq C,$$

where $C = C(\sup_M F, M, \omega)$.

We need the following lemma from [29].

Lemma 2.2. *Let (M, ω) be a compact Hermitian manifold of complex dimension n . If ϕ is a smooth solution of (1.1), then for any $\epsilon > 0$, we have*

$$\tilde{\Delta}(\Delta\phi) + (\epsilon - 1) \frac{|\tilde{\nabla}(\Delta\phi)|^2}{(n + \Delta\phi)} \geq \Delta F - A(1 + \frac{1}{\epsilon})(n + \Delta\phi)(n - \tilde{\Delta}\phi), \quad (2.1)$$

where $A = A(M, \omega, \|F\|_{L^\infty(M, \omega)})$.

Proof. We need the following equation from [29] (this equation is (9.5) in [29]).

$$\tilde{\Delta}(\log(tr_g \tilde{g})) \geq \frac{2}{(tr_g \tilde{g})^2} Re(\tilde{g}^{k\bar{l}} T_{ik}^i (tr_g \tilde{g})_{\bar{l}}) + \frac{\Delta F}{tr_g \tilde{g}} - C_1 tr_g \tilde{g} - C_1,$$

where the tensor T is the torsion of (M, ω) and $C_1 = C_1(M, \omega, \|F\|_{L^\infty(M, \omega)})$. By some calculation, we have

$$\tilde{\Delta}(\Delta\phi) - \frac{|\tilde{\nabla}(\Delta\phi)|^2}{(n + \Delta\phi)} \geq \frac{2}{(n + \Delta\phi)} Re(\tilde{g}^{k\bar{l}} T_{ik}^i (\Delta\phi)_{\bar{l}}) + \Delta F - C_2(n + \Delta\phi)(n - \tilde{\Delta}\phi),$$

where $C_2 = C_2(M, \omega, \|F\|_{L^\infty(M, \omega)})$ and we have used $tr_g \tilde{g} = (n - \tilde{\Delta}\phi) \geq ne^{-\frac{F}{n}}$. By the Cauchy-Schwarz inequality, for any $\epsilon > 0$, we get

$$\tilde{\Delta}(\Delta\phi) - \frac{|\tilde{\nabla}(\Delta\phi)|^2}{(n + \Delta\phi)} \geq -\epsilon \frac{|\tilde{\nabla}(\Delta\phi)|^2}{(n + \Delta\phi)} - \frac{A}{\epsilon}(n + \Delta\phi)(n - \tilde{\Delta}\phi) + \Delta F - A(n + \Delta\phi)(n - \tilde{\Delta}\phi),$$

where $A = A(M, \omega, \|F\|_{L^\infty(M, \omega)})$ and we have used $(n + \Delta\phi) \geq ne^{\frac{F}{n}}$. Then, we complete the proof. \square

Lemma 2.3. *Let (M, ω) be a compact Hermitian manifold of complex dimension n . If ϕ is a smooth solution of (1.1), then for any $p \geq 1$, we have*

$$\begin{aligned} \tilde{\Delta}(e^{f_p(\phi)}(n + \Delta\phi)^p) &\geq C_1(p)(n + \Delta\phi)^{p+\frac{1}{n-1}} - C_2(p)(n + \Delta\phi)^p \\ &\quad + pe^{f_p(\phi)}(n + \Delta\phi)^{p-1}\Delta F, \end{aligned}$$

where $f_p(\phi) = e^{-A(p+3)\phi}$, $C_1(p) = C_1(p, \|F\|_{L^\infty(M, \omega)}, M, \omega)$, $C_2(p) = C_2(p, \|F\|_{L^\infty(M, \omega)}, M, \omega)$ and $A = A(\|F\|_{L^\infty(M, \omega)}, M, \omega)$ (A is given in Lemma 2.2).

Proof. By direct calculation, we have

$$\begin{aligned}
\tilde{\Delta}(e^{f_p(\phi)}(n + \Delta\phi)^p) &= f'_p e^{f_p(\phi)}(\tilde{\Delta}\phi)(n + \Delta\phi)^p + (f_p'^2 + f_p'')e^{f_p(\phi)}|\tilde{\nabla}\phi|^2(n + \Delta\phi)^p \\
&\quad + p e^{f_p(\phi)}\tilde{\Delta}(\Delta\phi)(n + \Delta\phi)^{p-1} + p(p-1)e^{f_p(\phi)}|\tilde{\nabla}(\Delta\phi)|^2(n + \Delta\phi)^{p-2} \\
&\quad + 2p f'_p e^{f_p(\phi)}(n + \Delta\phi)^{p-1} \operatorname{Re}(\tilde{g}^{k\bar{l}}\phi_k(\Delta\phi)_{\bar{l}}).
\end{aligned} \tag{2.2}$$

By the definition of $f_p(\phi)$, we have

$$\begin{cases} f'_p(\phi) = -A(p+3)e^{-A(p+3)\phi} < 0 \\ f''_p(\phi) = A^2(p+3)^2 e^{-A(p+3)\phi} > 0 \end{cases}. \tag{2.3}$$

Thus, by the Cauchy-Schwarz inequality, we have

$$2\operatorname{Re}(\tilde{g}^{k\bar{l}}\phi_k(\Delta\phi)_{\bar{l}}) \leq \frac{(f_p'^2 + f_p'')(n + \Delta\phi)}{-p f'_p} |\tilde{\nabla}\phi|^2 + \frac{-p f'_p}{(f_p'^2 + f_p'')(n + \Delta\phi)} |\tilde{\nabla}(\Delta\phi)|^2,$$

which implies

$$\begin{aligned}
2p f'_p e^{f_p(\phi)}(n + \Delta\phi)^{p-1} \operatorname{Re}(\tilde{g}^{k\bar{l}}\phi_k(\Delta\phi)_{\bar{l}}) &\geq -(f_p'^2 + f_p'')e^{f_p(\phi)}|\tilde{\nabla}\phi|^2(n + \Delta\phi)^p \\
&\quad - \frac{p^2 f_p'^2}{f_p'^2 + f_p''} e^{f_p(\phi)}(n + \Delta\phi)^{p-2} |\tilde{\nabla}(\Delta\phi)|^2.
\end{aligned} \tag{2.4}$$

Combining (2.2) and (2.4), we have

$$\begin{aligned}
\tilde{\Delta}(e^{f_p(\phi)}(n + \Delta\phi)^p) &\geq f'_p e^{f_p(\phi)}(n + \Delta\phi)^p \tilde{\Delta}\phi + p e^{f_p(\phi)}\tilde{\Delta}(\Delta\phi)(n + \Delta\phi)^{p-1} \\
&\quad + |\tilde{\nabla}(\Delta\phi)|^2(n + \Delta\phi)^{p-2} e^{f_p(\phi)} \left(p(p-1) - \frac{p^2 f_p'^2}{f_p'^2 + f_p''} \right) \\
&\geq p e^{f_p(\phi)}(n + \Delta\phi)^{p-1} \left(\tilde{\Delta}(\Delta\phi) + \left(\frac{p f_p''}{(f_p')^2 + f_p''} - 1 \right) \frac{|\tilde{\nabla}(\Delta\phi)|^2}{(n + \Delta\phi)} \right) \\
&\quad + f'_p e^{f_p(\phi)}(n + \Delta\phi)^p \tilde{\Delta}\phi.
\end{aligned}$$

By Lemma 2.2 (take $\epsilon = \frac{p f_p''}{(f_p')^2 + f_p''}$), we obtain,

$$\begin{aligned}
\tilde{\Delta}(e^{f_p(\phi)}(n + \Delta\phi)^p) &\geq f'_p e^{f_p(\phi)}(n + \Delta\phi)^p \tilde{\Delta}\phi + p e^{f_p(\phi)}(n + \Delta\phi)^{p-1} \Delta F \\
&\quad - A p e^{f_p(\phi)}(n + \Delta\phi)^p (n - \tilde{\Delta}\phi) \left(1 + \frac{(f_p')^2 + f_p''}{p f_p''} \right) \\
&= n f'_p e^{f_p(\phi)}(n + \Delta\phi)^p + p e^{f_p(\phi)}(n + \Delta\phi)^{p-1} \Delta F \\
&\quad + e^{f_p(\phi)}(n + \Delta\phi)^p (n - \tilde{\Delta}\phi) \left(-f'_p - A p \left(1 + \frac{(f_p')^2 + f_p''}{p f_p''} \right) \right) \\
&\geq n f'_p e^{f_p(\phi)}(n + \Delta\phi)^p + p e^{f_p(\phi)}(n + \Delta\phi)^{p-1} \Delta F \\
&\quad + A e^{f_p(\phi)}(n + \Delta\phi)^p (n - \tilde{\Delta}\phi),
\end{aligned} \tag{2.5}$$

where we have used $\sup_M \phi = -1$ and (2.3). It is clear that

$$\text{tr}_g \tilde{g} \leq (\text{tr}_{\tilde{g}} g)^{n-1} \frac{\det \tilde{g}}{\det g},$$

which implies

$$(n + \Delta \phi) \leq (n - \tilde{\Delta} \phi)^{n-1} e^F. \quad (2.6)$$

Combining with (2.5) and (2.6), we complete the proof. \square

For convenience, we introduce a notation here, we define

$$u = e^{f_1(\phi)}(n + \Delta \phi). \quad (2.7)$$

Thus, by Young's inequality and Lemma 2.3, we have

$$\tilde{\Delta} u \geq e^{f_1(\phi)} \Delta F - \tilde{C}, \quad (2.8)$$

where $\tilde{C} = \tilde{C}(\|F\|_{L^\infty(M, \omega)}, M, \omega)$.

3 The Laplacian estimate

In this section, we remark that our constants may differ from line to line.

Lemma 3.1. *Let (M, ω) be a compact Hermitian manifold. If ϕ is a smooth solution of (1.1), then for any $f \in C^\infty(M)$, we have*

$$|\nabla f|^2 \leq C u |\tilde{\nabla} f|^2,$$

where u is defined by (2.7) and $C = C(\|F\|_{L^\infty(M, \omega)}, M, \omega)$.

Proof. By direct calculation, we have

$$|\nabla f|^2 \leq (n + \Delta \phi) |\tilde{\nabla} f|^2.$$

Combining with (2.7) and Theorem 2.1, we complete the proof. \square

Lemma 3.2. *Under the assumptions of Theorem 1.6, for any $p \geq 0$, we have*

$$\int_M |\nabla(u^{\frac{p}{2}})|^2 \omega^n \leq C(p^2 + 1) \int_M u^p (1 + |\nabla F|^2) \omega^n + C p \int_M u^p |\nabla \phi| |\nabla F| \omega^n + C \int_M u^{p+1} \omega^n,$$

where u is defined by (2.7) and $C = C(\|F\|_{L^\infty(M, \omega)}, M, \omega)$.

Proof. By Lemma 3.1 and direct calculation, we have

$$\begin{aligned} \int_M |\nabla(u^{\frac{p}{2}})|^2 \omega^n &\leq C_1 \int_M u |\tilde{\nabla}(u^{\frac{p}{2}})|^2 \tilde{\omega}^n \\ &= C_1 p \sqrt{-1} \int_M \partial u^p \wedge \bar{\partial} u \wedge \tilde{\omega}^{n-1} \\ &= -C_1 p \sqrt{-1} \int_M u^p \partial \bar{\partial} u \wedge \tilde{\omega}^{n-1} + \frac{C_1 p}{p+1} \sqrt{-1} \int_M \bar{\partial} u^{p+1} \wedge \partial \tilde{\omega}^{n-1} \\ &= -C_1 p \int_M u^p (\tilde{\Delta} u) \tilde{\omega}^n - \frac{C_1 p}{p+1} \sqrt{-1} \int_M u^{p+1} \partial \bar{\partial} \tilde{\omega}^{n-1}, \end{aligned}$$

where $C_1 = C_1(\|F\|_{L^\infty(M,\omega)}, M, \omega)$. Since M satisfies condition $(*)$ (when $n = 2$, all Hermitian manifolds satisfy the condition $(*)$), we have

$$-\frac{C_1 p}{p+1} \sqrt{-1} \int_M u^{p+1} \partial \bar{\partial} \tilde{\omega}^{n-1} \leq C_2 \int_M u^{p+1} \omega^n,$$

where $C_2 = C_2(\|F\|_{L^\infty(M,\omega)}, M, \omega)$. By (2.8) and $\tilde{\omega}^n = e^F \omega^n$, we compute

$$\begin{aligned} -C_1 p \int_M u^p (\tilde{\Delta} u) \tilde{\omega}^n &\leq C_3 p \int_M u^p \left(\tilde{C} - e^{f_1(\phi)} \Delta F \right) \tilde{\omega}^n \\ &\leq C_3 p \int_M u^p \tilde{\omega}^n - C_3 p \int_M e^{f_1(\phi)} u^p (\Delta(e^F) - e^F |\nabla F|^2) \omega^n \\ &\leq C_4 p \int_M u^p (1 + |\nabla F|^2) \omega^n + C_3 p \int_M \nabla(e^{f_1(\phi)} u^p) \nabla(e^F) \omega^n \\ &\quad - \sqrt{-1} C_3 p \int_M e^{f_1(\phi)} u^p \bar{\partial} e^F \wedge \partial \omega^{n-1}, \end{aligned}$$

where $C_3 = C_3(\|F\|_{L^\infty(M,\omega)}, M, \omega)$ and $C_4 = C_4(\|F\|_{L^\infty(M,\omega)}, M, \omega)$. It is clear that

$$\begin{aligned} C_3 p \int_M \nabla(e^{f_1(\phi)} u^p) \nabla(e^F) \omega^n &= C_3 p \int_M \nabla(e^{f_1(\phi)}) u^p \nabla(e^F) \omega^n + C_3 p \int_M e^{f_1(\phi)} \nabla(u^p) \nabla(e^F) \omega^n \\ &\leq C_5 p \int_M u^p |\nabla F| |\nabla \phi| \omega^n + \frac{1}{2} \int_M |\nabla u^{\frac{p}{2}}|^2 \omega^n + C_5 p^2 \int_M u^p |\nabla F|^2 \omega^n, \end{aligned}$$

where $C_5 = C_5(\|F\|_{L^\infty(M,\omega)}, M, \omega)$. Here we have used the Cauchy-Schwarz inequality. We notice that

$$-\sqrt{-1} C_3 p \int_M e^{f_1(\phi)} u^p \bar{\partial} e^F \wedge \partial \omega^{n-1} \leq C_6 p \int_M u^p |\nabla F| \omega^n,$$

where $C_6 = C_6(\|F\|_{L^\infty(M,\omega)}, M, \omega)$. Combining the above inequalities, we complete the proof. \square

Theorem 3.3. *Under the assumptions of Theorem 1.6, we have*

$$\|u\|_{L^\infty(M,\omega)} \leq C(\|u\|_{L^{\frac{q_0}{2}}(M,\omega)}, \|F\|_{W^{1,q_0}(M,\omega)}, q_0, M, \omega).$$

Proof. Without loss of generality, we can assume $q_0 < \infty$. We use the iteration method (see [24]).

By the Sobolev inequality (Corollary 5.2) and Lemma 3.1, for $p \geq 1$, we have

$$\begin{aligned} \left(\int_M u^{p\beta} \omega^n \right)^{\frac{1}{\beta}} &\leq C_1 \int_M u^p \omega^n + C_1 \int_M |\nabla(u^{\frac{p}{2}})|^2 \omega^n \\ &\leq C_1 \int_M u^p \omega^n + C_1 p^2 \int_M u^p (1 + |\nabla F|^2) \omega^n \\ &\quad + C_1 p \int_M u^p |\nabla \phi| |\nabla F| \omega^n + C_1 \int_M u^{p+1} \omega^n \\ &\leq C_1 p^2 \int_M u^{p+1} \omega^n + C_1 p^2 \int_M u^p |\nabla F|^2 \omega^n + C_1 p^2 \int_M u^p |\nabla \phi| |\nabla F| \omega^n, \end{aligned}$$

where $C_1 = C_1(\|F\|_{L^\infty(M,\omega)}, M, \omega)$. Here we have used Young's inequality and $p \leq p^2$. By the Hölder inequality, we have

$$\int_M u^p |\nabla F|^2 \omega^n \leq \left(\int_M u^{pr_0} \omega^n \right)^{\frac{1}{r_0}} \left(\int_M |\nabla F|^{q_0} \omega^n \right)^{\frac{2}{q_0}}$$

and

$$\int_M u^p |\nabla \phi| |\nabla F| \omega^n \leq \left(\int_M u^{pr_0} \omega^n \right)^{\frac{1}{r_0}} \left(\int_M |\nabla \phi|^{q_0} \omega^n \right)^{\frac{1}{q_0}} \left(\int_M |\nabla F|^{q_0} \omega^n \right)^{\frac{1}{q_0}},$$

where $\frac{1}{r_0} + \frac{2}{q_0} = 1$. Combining the above inequalities, when $pr_0 \geq p + 1$ (that is, $p \geq \frac{q_0-2}{2}$), we obtain

$$\begin{aligned} \|u\|_{L^{p\beta}(M,\omega)} &\leq (C_2 p^2 (\|\nabla \phi\|_{L^{q_0}(M,\omega)} + 1))^{\frac{1}{p}} \left(\|u\|_{L^{p+1}(M,\omega)}^{\frac{p+1}{p}} + \|u\|_{L^{pr_0}(M,\omega)} \right) \\ &\leq (C_2 p^2 (\|\nabla \phi\|_{L^{q_0}(M,\omega)} + 1))^{\frac{1}{p}} \|u\|_{L^{pr_0}(M,\omega)}^{\frac{p+1}{p}}, \end{aligned}$$

where $C_2 = C_2(\|F\|_{W^{1,q_0}(M,\omega)}, q_0, M, \omega)$. By Lemma 5.6, we have

$$\begin{aligned} \|\nabla \phi\|_{L^{q_0}(M,\omega)} &\leq C_3 \|u\|_{L^{\frac{2nq_0}{2n+q_0}}(M,\omega)} + C_3 \\ &\leq C_3 \|u\|_{L^{\frac{q_0}{2}}(M,\omega)} + C_3, \end{aligned}$$

where $C_3 = C_3(q_0, \|F\|_{\infty}, M, \omega)$. Thus, for any $k \geq 0$, we have

$$\|u\|_{L^{p_k\beta}(M,\omega)} \leq a_k \|u\|_{L^{p_k r_0}(M,\omega)}^{b_k}, \quad (3.1)$$

where

$$\begin{aligned} a_k &= \left(C_4 p_k^2 (\|u\|_{L^{\frac{q_0}{2}}(M,\omega)} + 1) \right)^{\frac{1}{p_k}}, \quad C_4 = C_4(\|F\|_{W^{1,q_0}(M,\omega)}, q_0, M, \omega) \\ b_k &= \frac{p_k + 1}{p_k} \text{ and } p_k = \frac{q_0 - 2}{2} \left(\frac{\beta}{r_0} \right)^k. \end{aligned}$$

By (3.1), we have

$$\|u\|_{L^{p_k\beta}(M,\omega)} \leq a_k a_{k-1}^{b_k} \cdots a_0^{b_k \cdots b_1} \|u\|_{L^{p_0 r_0}(M,\omega)}^{b_k \cdots b_0}. \quad (3.2)$$

Without loss of generality, we can assume that $a_k \geq 1$, for $k \geq 0$. We observe that $\prod_{i=0}^{\infty} b_k$ and $\prod_{i=0}^{\infty} a_k$ are convergent. In (3.2), let $k \rightarrow \infty$, we obtain

$$\|u\|_{L^{\infty}(M,\omega)} \leq C(\|u\|_{L^{\frac{q_0}{2}}(M,\omega)}, \|F\|_{W^{1,q_0}(M,\omega)}, q_0, M, \omega).$$

□

Lemma 3.4. *Under the assumptions of Theorem 1.6, for any $p \geq 1$, we have*

$$\int_M u^{p+\frac{1}{n-1}} \omega^n \leq C(p) \int_M u^{p-1} |\nabla \phi| |\nabla F| \omega^n + C(p) \int_M u^{p-1} |\nabla F|^2 \omega^n + C(p),$$

where $C(p) = C(p, \|F\|_{L^{\infty}(M,\omega)}, M, \omega)$.

Proof. By Lemma 2.3, we integrate on $(M, \tilde{\omega})$, then for any $p \geq 1$, we get

$$\begin{aligned} \int_M \tilde{\Delta}(e^{f_p(\phi)}(n + \Delta \phi)^p) \tilde{\omega}^n &\geq C_1(p) \int_M u^{p+\frac{1}{n-1}} \tilde{\omega}^n - C_2(p) \int_M u^p \tilde{\omega}^n \\ &\quad + p \int_M e^{f_p(\phi)}(n + \Delta \phi)^{p-1} \Delta F e^F \omega^n, \end{aligned}$$

where $C_1(p) = C_1(p, \|F\|_{L^\infty(M, \omega)}, M, \omega)$ and $C_2(p) = C_2(p, \|F\|_{L^\infty(M, \omega)}, M, \omega)$. Here we have used (2.7) and Theorem 2.1. Since M satisfies the condition $(*)$ (when $n = 2$, all Hermitian manifolds satisfy the condition $(*)$), we have

$$\begin{aligned} \int_M \tilde{\Delta}(e^{f_p(\phi)}(n + \Delta\phi)^p) \tilde{\omega}^n &= n \int_M e^{f_p(\phi)}(n + \Delta\phi)^p \sqrt{-1} \partial \bar{\partial} \tilde{\omega}^{n-1} \\ &\leq C_3(p) \int_M u^p \omega^n, \end{aligned}$$

where $C_3(p) = C_3(p, \|F\|_{L^\infty(M, \omega)}, M, \omega)$. Combining the above inequalities, we compute

$$\begin{aligned} \int_M u^{p+\frac{1}{n-1}} \omega^n &\leq C_4(p) \int_M e^{f_p(\phi)}(n + \Delta\phi)^{p-1} (|\nabla F|^2 e^F - \Delta(e^F)) \omega^n + C_5(p) \int_M u^p \omega^n \\ &\leq C_5(p) \int_M u^{p-1} |\nabla F|^2 \omega^n + C_4(p) \int_M \nabla \left(e^{f_p(\phi)}(n + \Delta\phi)^{p-1} \right) \nabla F e^F \omega^n \\ &\quad - C_4(p) \sqrt{-1} \int_M e^{f_p(\phi)}(n + \Delta\phi)^{p-1} \bar{\partial} e^F \wedge \partial \omega^{n-1} + C_5(p) \int_M u^p \omega^n \\ &\leq C_5(p) \int_M u^p \omega^n + C_5(p) \int_M u^{p-1} |\nabla F|^2 \omega^n + C_5(p) \int_M u^{p-1} |\nabla F| \omega^n \\ &\quad + C_5(p) \int_M |\nabla(u^{p-1})| |\nabla F| \omega^n + C_5(p) \int_M u^{p-1} |\nabla \phi| |\nabla F| \omega^n, \end{aligned}$$

where $C_4(p) = C_4(p, \|F\|_{L^\infty(M, \omega)}, M, \omega)$ and $C_5(p) = C_5(p, \|F\|_{L^\infty(M, \omega)}, M, \omega)$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} C_5(p) \int_M |\nabla(u^{p-1})| |\nabla F| \omega^n &= C_5(p) \int_M |\nabla(u^{\frac{p-1}{2}})| u^{\frac{p-1}{2}} |\nabla F| \omega^n \\ &\leq C_5(p) \int_M |\nabla(u^{\frac{p-1}{2}})|^2 \omega^n + C_5(p) \int_M u^{p-1} |\nabla F|^2 \omega^n. \end{aligned}$$

Combining with the above inequalities and Lemma 3.2, we get

$$\int_M u^{p+\frac{1}{n-1}} \omega^n \leq C_6(p) \int_M u^p \omega^n + C_6(p) \int_M u^{p-1} |\nabla \phi| |\nabla F| \omega^n + C_6(p) \int_M u^{p-1} |\nabla F|^2 \omega^n,$$

where $C_6(p) = C_6(p, \|F\|_{L^\infty(M, \omega)}, M, \omega)$. By Young's inequality, we complete the proof. \square

Now, we are in the position to prove Theorem 1.6.

Proof of Theorem 1.6. Without loss of generality, we assume that $q_0 < \infty$. By Lemma 3.4 and $F \in W^{1, q_0}$, for any $p \geq 1$, we have

$$\begin{aligned} \int_M u^{p+\frac{1}{n-1}} \omega^n &\leq C_1(p) \int_M u^{p-1} |\nabla \phi| |\nabla F| \omega^n + C_1(p) \int_M u^{p-1} |\nabla F|^2 \omega^n + C_1(p) \\ &\leq C_1(p) \int_M u^{p-1} |\nabla \phi|^2 \omega^n + C_2(p) \int_M u^{(p-1)\frac{q_0}{q_0-2}} \omega^n + C_2(p), \end{aligned}$$

where $C_1(p) = C_1(p, \|F\|_{L^\infty(M, \omega)}, M, \omega)$, $C_2(p) = C_2(p, \|F\|_{W^{1, q_0}(M, \omega)}, q_0, M, \omega)$ and we have used the Hölder inequality in the last line. When p satisfies the following condition

$$p + \frac{1}{n-1} > (p-1) \frac{q_0}{q_0-2} \Leftrightarrow p < \frac{q_0-2}{2n-2} + \frac{q_0}{2}$$

and $p \geq 1$, we can use Young's inequality to get the following inequality

$$\int_M u^{p+\frac{1}{n-1}} \omega^n \leq C_3(p) \int_M u^{p-1} |\nabla \phi|^2 \omega^n + C_3(p),$$

where $C_3(p) = C_3(p, \|F\|_{W^{1, q_0}(M, \omega)}, q_0, M, \omega)$. Now, we take $p = \frac{q_0}{2} - \frac{1}{n-1}$, we obtain

$$\begin{aligned} \int_M u^{\frac{q_0}{2}} \omega^n &\leq C_4 \int_M u^{\frac{q_0}{2}-\beta} |\nabla \phi|^2 \omega^n + C_4 \\ &\leq \frac{1}{2} \int_M u^{(\frac{q_0}{2}-\beta) \frac{q_0}{q_0-2\beta}} \omega^n + C_4 \int_M |\nabla \phi|^{\frac{q_0}{\beta}} \omega^n + C_4, \end{aligned}$$

where $C_4 = C_4(\|F\|_{W^{1, q_0}(M, \omega)}, q_0, M, \omega)$ and $\beta = \frac{n}{n-1}$. It then follows that

$$\|u\|_{L^{\frac{q_0}{2}}(M, \omega)} \leq C_4 \|\nabla \phi\|_{L^{\frac{q_0}{\beta}}(M, \omega)}^{\frac{2}{\beta}} + C_4. \quad (3.3)$$

By Lemma 5.7, we have

$$\|\nabla \phi\|_{L^{\frac{q_0}{\beta}}(M, \omega)} \leq C_5 \|u\|_{L^{\frac{q_0}{2\beta}}(M, \omega)}^{\frac{1}{2}} + C_5, \quad (3.4)$$

where $C_5 = C_5(q_0, \|F\|_{L^\infty(M, \omega)}, M, \omega)$. Combining (3.3), (3.4) and $\beta > 1$, we get

$$\|u\|_{L^{\frac{q_0}{2}}(M, \omega)} \leq C_6(\|F\|_{W^{1, q_0}(M, \omega)}, q_0, M, \omega).$$

By Theorem 3.3, we complete the proof. \square

4 The Hölder estimate of second order and solve the equation

We note that, when F is in W^{1, q_0} for any $q_0 > 2n$, Sobolev embedding implies that $F \in C^{\alpha_0}$, where $\alpha_0 = 1 - \frac{2n}{q_0}$. By Theorem 1.1 in [30], we have the following theorem.

Theorem 4.1. *Let (M, ω) be a compact Hermitian manifold. If ϕ is a smooth solution of (1.1) and $F \in C^{\alpha_0}$, then there exists a constant $\alpha \in (0, 1)$ such that*

$$\|\phi\|_{C^{2, \alpha}(M, \omega)} \leq C,$$

where α and C depend only on $\|\phi\|_{L^\infty(M, \omega)}$, $\|\Delta \phi\|_{L^\infty(M, \omega)}$, α_0 , $\|F\|_{C^{\alpha_0}(M, \omega)}$, q_0 , M and ω .

Now we are in the position to prove Theorem 1.7.

Proof of Theorem 1.7. Our argument here is similar to the argument in [10]. If $F \in W^{1, q_0}$ on M such that $\|F\|_{W^{1, q_0}(M, \omega)} \leq \Lambda$ for some positive constant Λ . Let $\{F_k\}$ be a sequence of smooth functions

such that $F_k \rightarrow F$ in W^{1,q_0} . In particular, we can assume $\|F_k\|_{W^{1,q_0}(M,\omega)} \leq \Lambda + 1$ for any k . By [28], there is a unique smooth solution ϕ_k and constant b_k such that

$$\det(g_{i\bar{j}} + (\phi_k)_{i\bar{j}}) = e^{F+b_k} \det(g_{i\bar{j}}),$$

such that $(g_{i\bar{j}} + (\phi_k)_{i\bar{j}}) > 0$ with normalized condition $\sup_M \phi_k = -1$. By Maximum Principle, we have

$$|b_k| \leq C_1(\|F_k\|_{L^\infty(M,\omega)}, M, \omega). \quad (4.1)$$

By Theorem 1.6, Theorem 2.1 and Theorem 4.1, there exists a constant $\alpha \in (0, 1)$ such that

$$\|\phi_k\|_{C^{2,\alpha}(M,\omega)} \leq C_2(\|F_k\|_{W^{1,q_0}(M,\omega)}, q_0, M, \omega).$$

To get W^{3,q_0} -estimate, we can localize the estimate as follows. Let ∂ denote an arbitrary first order differential operator in a domain $\Omega \subset M$. Since we have $C^{2,\alpha}$ -estimate, we compute in Ω

$$\tilde{\Delta}_{g_k}(\partial\phi_k) = \partial(F_k + \log(\det(g_{i\bar{j}}))) - (g_k)^{i\bar{j}} \partial g_{i\bar{j}},$$

where $(g_k)_{i\bar{j}} = g_{i\bar{j}} + (\phi_k)_{i\bar{j}}$. Since $\tilde{\Delta}_{g_k}$ is a uniform elliptic operator, by L^p estimates (for example, see [17]), for any $\Omega' \subset \Omega$, we have

$$\|\partial\phi_k\|_{W^{2,q_0}(\Omega',\omega)} \leq C_3(\Omega, \Omega', q_0, \Lambda, \omega),$$

which implies

$$\|\phi_k\|_{W^{3,q_0}(M,\omega)} \leq C_4(\|F\|_{W^{1,q_0}(M,\omega)}, q_0, \Lambda, M, \omega). \quad (4.2)$$

By (4.1) and (4.2), we know that there is a subsequence $\{(\phi_{k_l}, b_{k_l})\}$ of $\{(\phi_k, b_k)\}$ such that $\{b_{k_l}\}$ converges to b and $\{\phi_{k_l}\}$ converges to $\phi \in W^{3,q_0}$ such that $(g_{i\bar{j}} + \phi_{i\bar{j}}) > 0$, which defines a W^{1,q_0} Hermitian metric. Hence ϕ with constant b is a classical solution of the complex Monge-Ampère equation. The uniqueness follows from Remark 5.1 in [27]. \square

5 Appendix

Let $g_{\mathbb{R}}$ denote the Riemannian metric induced by g , thus $(M, g_{\mathbb{R}})$ is a Riemannian manifold of real dimension $2n$. In Appendix, we deduce some interpolation inequalities on Hermitian manifold (M, ω) by using some fundamental inequalities on Riemannian manifold $(M, g_{\mathbb{R}})$.

Let us recall the definition of $g_{\mathbb{R}}$ first. For any local holomorphic coordinates (z^1, \dots, z^n) with $z^i = x^i + \sqrt{-1}y^i$, $(x^1, \dots, x^n, y^1, \dots, y^n)$ form a smooth local coordinates. We define,

$$g_{\mathbb{R}}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{\mathbb{R}}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 2\Re(g_{i\bar{j}})$$

while

$$g_{\mathbb{R}}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) = 2\Im(g_{i\bar{j}}).$$

For Riemannian metric $g_{\mathbb{R}}$, let $\nabla_{\mathbb{R}}$ and $dV_{\mathbb{R}}$ denote the Levi-Civita connection and the volume form, respectively. By direct calculation, we have

$$dV_{\mathbb{R}} = \frac{1}{n!} \omega^n. \quad (5.1)$$

For convenience, we introduce some notations. For any function $f \in C^\infty(M)$, let $\nabla_{\mathbb{R}}^m f$ and $\Delta_{\mathbb{R}} f$ denote the m^{th} covariant derivative and the Laplacian of f with respect to $g_{\mathbb{R}}$. Let $\|f\|_{L^p(M, g_{\mathbb{R}})}$ and $\|\nabla_{\mathbb{R}}^m f\|_{L^p(M, g_{\mathbb{R}})}$ denote the corresponding norms with respect to $(M, g_{\mathbb{R}})$.

Thus, by (5.1) and some calculations, we have the following lemma.

Lemma 5.1. *For any $f \in C^\infty(M)$, we have*

$$\|f\|_{L^p(M, g_{\mathbb{R}})} = C_1(p) \|f\|_{L^p(M, \omega)}, \quad \|\nabla_{\mathbb{R}} f\|_{L^p(M, g_{\mathbb{R}})} = C_2(p) \|\nabla f\|_{L^p(M, \omega)},$$

where $C_1(p) = C_1(p, n)$ and $C_2(p) = C_2(p, n)$.

Corollary 5.2. *For any $f \in C^\infty(M)$, we have Sobolev inequality*

$$\left(\int_M f^{2\beta} \omega^n \right)^{\frac{1}{\beta}} \leq C \int_M f^2 \omega^n + C \int_M |\nabla f|^2 \omega^n,$$

where $\beta = \frac{n}{n-1}$ and $C = C(M, \omega)$.

Proof. By Sobolev embedding $W^{1,2}(M, g_{\mathbb{R}}) \hookrightarrow L^{2\beta}(M, g_{\mathbb{R}})$, we have

$$\left(\int_M f^{2\beta} dV_{\mathbb{R}} \right)^{\frac{1}{\beta}} \leq C_s \int_M f^2 dV_{\mathbb{R}} + C_s \int_M |\nabla_{\mathbb{R}} f|^2 dV_{\mathbb{R}},$$

where $C_s = C_s(M, g_{\mathbb{R}})$. Thus, combining with Lemma 5.1, we complete the proof. \square

Because $(M, g_{\mathbb{R}})$ is a Riemannian manifold of real dimension $2n$. So we have the following interpolation inequality (for example, see [1]).

Theorem 5.3. *Let q, r be real numbers $1 \leq q, r \leq +\infty$ and j, m integers $0 \leq j < m$. Then there exists a constant*

$$C = C(M, g_{\mathbb{R}}, m, j, q, r, \alpha)$$

such that for all $f \in C^\infty(M)$ with $\int_M f dV_{\mathbb{R}} = 0$, we have

$$\|\nabla_{\mathbb{R}}^j f\|_{L^p(M, g_{\mathbb{R}})} \leq C \|\nabla_{\mathbb{R}}^m f\|_{L^r(M, g_{\mathbb{R}})}^\alpha \|f\|_{L^q(M, g_{\mathbb{R}})}^{1-\alpha}, \quad (5.2)$$

where

$$\frac{1}{p} = \frac{j}{2n} + \alpha \left(\frac{1}{r} - \frac{m}{2n} \right) + (1 - \alpha) \frac{1}{q}$$

for all α in the interval $\frac{j}{m} \leq \alpha \leq 1$, for which p is non-negative. If $r = \frac{2n}{m-j} \neq 1$, then (5.2) is not valid for $\alpha = 1$.

Corollary 5.4. *Let $f \in C^\infty(M)$, for any $\epsilon > 0$ and $1 \leq p < \infty$, we have*

$$\|\nabla_{\mathbb{R}} f\|_{L^p(M, g_{\mathbb{R}})} \leq \epsilon \|\nabla_{\mathbb{R}}^2 f\|_{L^p(M, g_{\mathbb{R}})} + C(\epsilon, p) \|f\|_{L^p(M, g_{\mathbb{R}})},$$

where $C(\epsilon, p) = C(\epsilon, p, M, \omega)$.

Proof. Define $\tilde{f} = f - \frac{1}{\text{Vol}(M, g_{\mathbb{R}})} \int_M f dV_{\mathbb{R}}$, then $\int_M \tilde{f} dV_{\mathbb{R}} = 0$. By Theorem 5.3, we have

$$\|\nabla_{\mathbb{R}} \tilde{f}\|_{L^p(M, g_{\mathbb{R}})} \leq C_1(p) \|\nabla_{\mathbb{R}}^2 \tilde{f}\|_{L^p(M, g_{\mathbb{R}})}^{\frac{1}{2}} \|\tilde{f}\|_{L^p(M, g_{\mathbb{R}})}^{\frac{1}{2}},$$

where $C_1(p) = C_1(p, M, g_{\mathbb{R}})$. Thus, by the Cauchy-Schwarz inequality, for any $\epsilon > 0$, we obtain

$$\|\nabla_{\mathbb{R}} \tilde{f}\|_{L^p(M, g_{\mathbb{R}})} \leq \epsilon \|\nabla_{\mathbb{R}}^2 \tilde{f}\|_{L^p(M, g_{\mathbb{R}})} + C_2(\epsilon, p) \|\tilde{f}\|_{L^p(M, g_{\mathbb{R}})},$$

where $C_2(\epsilon, p) = C_2(\epsilon, p, M, g_{\mathbb{R}})$. By the definition of \tilde{f} , we complete the proof. \square

Lemma 5.5. *Let (M, ω) be a compact Hermitian manifold of complex dimension n . If ϕ is a smooth solution of (1.1), then for any $1 < p < \infty$, we have*

$$\|\Delta_{\mathbb{R}} \phi\|_{L^p(M, \omega)} \leq C_1(p) \|\Delta \phi\|_{L^p(M, \omega)} + C_2(p),$$

where $C_1 = C_1(p, n)$ and $C_2(p) = C_2(p, \|F\|_{L^\infty(M, \omega)}, M, \omega)$.

Proof. By some calculations, we have

$$\|\Delta_{\mathbb{R}} \phi\|_{L^p(M, g_{\mathbb{R}})} \leq 2 \|\Delta \phi\|_{L^p(M, g_{\mathbb{R}})} + C_3(p) \|\nabla_{\mathbb{R}} \phi\|_{L^p(M, g_{\mathbb{R}})}, \quad (5.3)$$

where $C_3 = C_3(p, M, \omega)$. For (5.3), one can find more details in [26] (Lemma 3.2 in [26] shows the exact relation between $\Delta_{\mathbb{R}}$ and 2Δ). By Corollary 5.4, we obtain

$$C_3(p) \|\nabla_{\mathbb{R}} \phi\|_{L^p(M, g_{\mathbb{R}})} \leq \frac{1}{2} \|\Delta_{\mathbb{R}} \phi\|_{L^p(M, g_{\mathbb{R}})} + C_4(p) \|\phi\|_{L^p(M, g_{\mathbb{R}})}, \quad (5.4)$$

where $C_4 = C_4(p, M, \omega)$. Combining with (5.3) and (5.4), we obtain

$$\|\Delta_{\mathbb{R}} \phi\|_{L^p(M, g_{\mathbb{R}})} \leq 4 \|\Delta \phi\|_{L^p(M, g_{\mathbb{R}})} + C_5(p) \|\phi\|_{L^p(M, g_{\mathbb{R}})},$$

where $C_5 = C_5(p, M, \omega)$. By Theorem 2.1 and Lemma 5.1, we complete the proof. \square

Lemma 5.6. *Under the assumptions of Theorem 1.6, for any $1 < p < 2n$, we have*

$$\|\nabla \phi\|_{L^{\frac{2np}{2n-p}}(M, \omega)} \leq C(p) \|u\|_{L^p(M, \omega)} + C(p),$$

where u is defined by (2.7) and $C(p) = C(p, \|F\|_{L^\infty(M, \omega)}, M, \omega)$.

Proof. By Sobolev embedding $W^{2,p}(M, g_{\mathbb{R}}) \hookrightarrow W^{1, \frac{2np}{2n-p}}(M, g_{\mathbb{R}})$, we have

$$\|\nabla_{\mathbb{R}} \phi\|_{L^{\frac{2np}{2n-p}}(M, g_{\mathbb{R}})} \leq C_1(p) \|\nabla_{\mathbb{R}}^2 \phi\|_{L^p(M, g_{\mathbb{R}})} + C_1(p) \|\nabla_{\mathbb{R}} \phi\|_{L^p(M, g_{\mathbb{R}})} + C_1(p) \|\phi\|_{L^p(M, g_{\mathbb{R}})},$$

where $C_1(p) = C_1(p, M, g_{\mathbb{R}})$. Combining with Corollary 5.4, we have

$$\|\nabla \phi\|_{L^{\frac{2np}{2n-p}}(M, g_{\mathbb{R}})} \leq C_2(p) \|\nabla_{\mathbb{R}}^2 \phi\|_{L^p(M, g_{\mathbb{R}})} + C_2(p) \|\phi\|_{L^p(M, g_{\mathbb{R}})},$$

where $C_2(p) = C_2(p, M, g_{\mathbb{R}})$. By Theorem 2.1 and L^p estimates (for example, see [17]), we have

$$\|\nabla \phi\|_{L^{\frac{2np}{2n-p}}(M, g_{\mathbb{R}})} \leq C_3(p) \|\Delta_{\mathbb{R}} \phi\|_{L^p(M, g_{\mathbb{R}})} + C_3(p),$$

where $C_3(p) = C_3(p, \|F\|_{L^\infty(M, \omega)}, M, g_{\mathbb{R}})$. By Lemma 5.1 and Lemma 5.5, we have

$$\|\nabla \phi\|_{L^{\frac{2np}{2n-p}}(M, \omega)} \leq C_4(p) \|\Delta \phi\|_{L^p(M, \omega)} + C_4(p),$$

where $C_4(p) = C_4(p, \|F\|_{L^\infty(M, \omega)}, M, g_{\mathbb{R}})$. By (2.7) and Theorem 2.1, we complete the proof. \square

Lemma 5.7. *Let p, r be real numbers $1 < p, r < \infty$. Under the assumptions of Theorem 1.6, we have*

$$\|\nabla\phi\|_{L^p(M,\omega)} \leq C(p, r)\|u\|_{L^r}^\alpha + C(p, r),$$

where $C(p, r) = C(p, r, \|F\|_{L^\infty(M,\omega)}, M, \omega)$ and

$$\frac{1}{p} = \frac{1}{2n} + \alpha\left(\frac{1}{r} - \frac{1}{n}\right),$$

for α in the $\frac{1}{2} \leq \alpha < 1$.

Proof. Define $\tilde{\phi} = \phi - \frac{1}{\text{Vol}(M, g_R)} \int_M \phi dV_R$, then $\int_M \tilde{\phi} dV_R = 0$, by Theorem 2.1, Lemma 5.1 and Theorem 5.3, we have

$$\|\nabla_R \tilde{\phi}\|_{L^p(M, g_R)} \leq C_1(p, r) \|\nabla_R^2 \tilde{\phi}\|_{L^r(M, g_R)}^\alpha$$

which implies

$$\|\nabla_R \phi\|_{L^p(M, g_R)} \leq C_1(p, r) \|\nabla_R^2 \phi\|_{L^r(M, g_R)}^\alpha,$$

where $C_1(p, r) = C_1(p, r, \|F\|_{L^\infty(M,\omega)}, M, \omega)$ and $\alpha = \frac{(2n-p)r}{(2n-2r)p}$. Combining Lemma 5.1, Lemma 5.5, (2.7) and L^p estimates (for example, see [17]), we complete the proof. \square

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